Introduction

• The whole purpose of science is to find meaningful simplicity in the midst of disorderly complexity.


A hierarchy is always associated with a network. Batty and Longley (1994) once pointed out that a hierarchy and a network represent different sides of the same coin.


The dummy hierarchy is in fact a mathematical transform rather than a physical structure.

• The mathematical transform can be termed “hierarchical scaling and rescaling (HSR)”. 

I try to develop a theory and method of hierarchical scaling and rescaling based on urban studies.

• However, only a little people in the world understand my idea of dummy network and hierarchical rescaling.
Introduction

• The fewer the people who understand my theory are, the more important maybe it is because this may suggest that the theory goes before.

• My theory of hierarchy has a novel framework, but it comprises many traditional theoretical elements such as Zipf’s law, the alloemtry growth law, Pareto distribution, and $1/f$ noise.

• Many people can see the traditional elements, but fail to see the novel framework.

Introduction

• The principal functions and uses of the hierarchical scaling and rescaling are as follows.

• First, it provides a new way of looking at the scaling laws and complex systems.

• Second, it provides more than three approaches to estimating the scaling exponent.

• The scaling exponent values depend on algorithms (e.g. the least squares method, the maximum likelihood method).

Introduction

• By hierarchical rescaling, the power-law method, the exponential-law method, and the common ratio method can be employed at one time to work out the scaling exponent.

• By comparing the results from different methods, we can judge which algorithm and result is the best.

• Third, it can be employ to resolve many scientific problems.

• In geography, for example, it can be used to testify central place theory, to estimate the distance friction coefficient, to calculate the allometric scaling exponent, to develop the optimization theory of space filling, and so on.
Spatial disaggregation

• (1) Strict subdivision
• (2) Hierarchy
• (3) Network structure

by Batty and Longley, 1994, Page 45

Spatial disaggregation

• (1) Strict subdivision—Recurrence
• (2) Hierarchy—Self-similar
• (3) Network structure—Self-similar and scaling invariance

• Power laws ↔ Scaling laws
• A power law is always a special form of scaling laws.

Scaling law

• In mathematics, an eigenfunction of an operator, \( T \), defined in some function space is any non-zero function \( f \) in that space that returns from the operator exactly as is, except for a multiplicative scaling factor \( \lambda \).

\[
Tf(x) = \lambda f(x)
\]

\( T \) – transform
\( f(x) \) – eigenfunction
\( \lambda \) – eigenvalue

Scaling law

• Let’s take an exponential function to show what is an eigenfunction.

\[
f(x) = Ae^{-bx}
\]

Scaling law

• Define a differential operator such as

\[
D = \frac{d}{dx}
\]
Scaling law
• Thus we have:
  \[Df(x) = \frac{d}{dx}(Ae^{-bx}) = -bAe^{-bx} = \lambda f(x)\]
  \[\lambda = -b\]

Scaling law
• If the transform is a linear operator indicating scaling transform (dilation transform), the eigenfunction will suggest a scaling law.
• For example, Zipf's law.

Scaling law
\[P(k) = P_1 k^{-q}\]
• Zipf's law.
  - \(k\) – rank
  - \(P\) – size
  - \(P_1\) – coefficient
  - \(q\) – scaling exponent

Scaling law
• Define an operator indicative of scaling transform \(T\), we have
  \[T[P(k)] = P(\xi k) = P_1 (\xi k)^{-q}\]
  \[= \xi^{-q} (P_1 k^{-q}) = \xi^{-q} P(k) = \lambda P(k)\]
  \[\lambda = \xi^{-q}\]

Scaling law
• This suggests that Zipf's law is a scaling law, and the scaling exponent is associated with the eigenvalue.

Scaling law
• Generally, we have a scaling relation
  \[f(\xi x) = \xi^{\pm a} f(x)\]
  - \(\xi\) – scale factor
  - \(a\) – scaling exponent
**Scaling law**

- The solution to the functional equation is always a scaling law.
- The invariance under some transform (change) is a kind of symmetry.
- The scaling law implies dilation symmetry.

**Example**

For example, for fractals, according to B.B. Mandelbrot (1989), “It is based on a form of symmetry that had previously been underused, namely invariance under contraction or dilation.”


**Geographical spatial information:**

- Scaling—Symmetry—Spatial complexity
- Scaling analysis, spectral analysis, and spatial correlation analysis.

**Hierarchical scaling**

- If the scaling exponent $q=1$, Zipf’s law suggests the rank-size rule.
- The rank-size rule is the strong form of Zipf’s law.

$$P(k) = \frac{P_1}{k}$$

$p_1 = 1$ unit, $k = 1, 2, 3, \ldots$

**Zipf’s law**

$$P(k) = P_1 k^{-q}$$

The rank—size rule

$$P(k) = \frac{P_1}{k} = \frac{1}{k}$$

$p_1 = 1$, $q = 1$, $k = 1, 2, 3, \ldots$

**Hierarchical rescaling**

- A rank-size scaling can be transformed into a hierarchical scaling.
- I call this process “hierarchical rescaling”.
- This is important and useful scaling method.
- But little people know it significance.
Hierarchical rescaling

• A self-similar hierarchy with cascade structure (The first four classes).

Hierarchical rescaling

• The rank-size rule can be abstracted as harmonic sequence.
• Let \( P_1 = 1 \), the harmonic sequence is as follows:
  \[ \{1, 1/2, 1/3, 1/4, \ldots, 1/k, \ldots\} \]

Hierarchical rescaling

• The harmonic sequence can be rearranged in terms of the self-similar hierarchy as below:

\[
\begin{align*}
m = 1: & 1 \\
m = 2: & \frac{1}{2}, \frac{1}{3} \\
m = 3: & \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7} \\
m = 4: & \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15} \\
& \vdots
\end{align*}
\]

Hierarchical rescaling

• City number in different classes \( f_m \) forms a geometric sequence such as \( \{1, 2, 4, 8, 16, \ldots, 2^{m-1}, \ldots\} \), where \( m \) is the sequence number of classes.
• The total population approaches \( S_m = \ln(2) \approx 0.6931 \) unit.
• The average size is \( P_m = \frac{S_m}{f_m} = \frac{\ln(2)}{2^{m-1}} \).

Hierarchical rescaling

• There are two important common ratios—number ratio \( r_f \) and size ratio \( r_p \):

\[
\begin{align*}
r_f &= f_{m+1} / f_m \\
r_p &= P_m / P_{m+1}
\end{align*}
\]

<table>
<thead>
<tr>
<th>Level (m)</th>
<th>City number ( f_m )</th>
<th>City sum ( (S_m = f_m P_m) )</th>
<th>Average size ( (P_m) )</th>
<th>Size ratio ( (P_m / P_{m+1}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>128</td>
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<td>0.0054</td>
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<td>256</td>
<td>0.6941</td>
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</tr>
<tr>
<td>10</td>
<td>512</td>
<td>0.6936</td>
<td>0.0014</td>
<td>2.000</td>
</tr>
<tr>
<td>( M )</td>
<td>( 2^{M+1} )</td>
<td>( \ln(2) )</td>
<td>( \ln(2)(2^{M+1}) )</td>
<td>2.000</td>
</tr>
</tbody>
</table>
Hierarchical rescaling

Thus a pair of exponential relations can be given as follows:

\[ \begin{align*}
  f_m &= f_f r_f^{m-1} = 2^{m-1} \\
  P_m &= P_p r_p^{1-m} = \ln(2) \cdot 2^{1-m}
\end{align*} \]

From the pair of exponential laws follows a hierarchical scaling law in the form:

\[ \begin{align*}
  f_m &= \mu P_m^D \\
  \mu &= f_1 P_1^D = 1 \\
  D &= \frac{\ln r_f}{\ln r_p} = 1
\end{align*} \]

The harmonic sequence can also be rearranged as follows:

\begin{align*}
  m = 1: & \quad 1 \\
  m = 2: & \quad \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \frac{9}{32}, \frac{11}{64}, \frac{13}{128}, \ldots \\
  m = 3: & \quad \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}, \frac{1}{17}, \frac{1}{19}, \frac{1}{21}, \frac{1}{23}, \ldots \\
  m = 4: & \quad \frac{1}{4}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \frac{1}{18}, \frac{1}{19}, \frac{1}{20}, \frac{1}{21}, \frac{1}{22}, \frac{1}{23}, \frac{1}{24}, \frac{1}{25}, \frac{1}{26}, \ldots \\
  \ldots
\end{align*} \]

The self-similar hierarchy with cascade structure is as below:

Class 1

Class 2

Class 3

Class 4

1/14, 1/15, 1/16, 1/17, 1/18, 1/19, 1/20, 1/21, 1/22, 1/23, 1/24, 1/25, 1/26, 1/27, 1/28, 1/29, 1/30, 1/31, 1/32, 1/33, 1/34, 1/35, 1/36, 1/37, 1/38, 1/39, 1/40

The final results are a pair of exponential laws or a hierarchical power law:

\[ \begin{align*}
  f_m &= f_f r_f^{m-1} = 3^{m-1} \\
  P_m &= P_p r_p^{1-m} = \ln(3) \cdot 3^{1-m}
\end{align*} \]

Power law

\[ f_m = \mu P_m^D \]

<table>
<thead>
<tr>
<th>Rank-size power law</th>
<th>Exponential laws</th>
<th>Hierarchical power law</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complexity</td>
<td>Simplicity</td>
<td>Complexity</td>
</tr>
<tr>
<td>Scaling symmetry</td>
<td>Translational symmetry</td>
<td>Scaling symmetry</td>
</tr>
<tr>
<td>Complexity signature</td>
<td>Physical understanding</td>
<td>Computation &amp; derivation</td>
</tr>
</tbody>
</table>
Hierarchical rescaling

• A geometric subdivision theorem of the harmonic sequence can be proved as follows: “If we group a harmonic sequence \( \{1/k\} \) into different classes, and the amount of numbers in each class form a geometric sequence such as \( N^0, N^1, N^2, \ldots, N^m \), then the sum of numerical value in each class asymptotically approaches a constant \( \ln N \).”

Evidences

• Recently, Bin Jiang and his coworkers have proposed a concept of “natural city” and developed a novel approach to measuring objective city sizes based on street nodes or blocks and thus urban boundaries can be naturally defined (http://arxiv.org/find/all).

Evidences

• The street nodes are defined as street intersections and ends, while the naturally defined urban boundaries constitute what is called natural cities.
• The street nodes are significantly correlated with population of cities as well as city areal extents.

Evidences

• The city data are extracted from massive volunteered geographic information OpenStreetMap databases through some data-intensive computing processes and three data sets on cities of France, Germany, and the United Kingdom (UK) are formed.

Evidences

• Among all the three data sets, the set for Germany is the largest one, which encompasses the 5160 natural cities.
• Let’s take Germany cities as an example to illustrate the inherent relation between the rank-size rule and the \( N^a \) principle.

Evidences

• For the natural cities, the population size measurement \( (P) \) should be replaced by the amount of blocks in the physical areal extent \( (A) \), which can be treated as a new size measurement of cities.
Empirically, the largest 5160 natural cities and towns follow the rank-size rule and we have
\[ \hat{P}_k = 160175.044 k^{-1.051} \]

The goodness of fit is about \( R^2=0.993 \), and the scaling exponent is about \( q=1.051 \), which is close to 1.

The rank-size distribution follows the power law.

However, it is easy to be mistaken as a lognormal distribution by its form and the goodness of fit (\( R^2 \)) values of regression analysis.

Three kinds of hierarchies are constructed as follows.

Taking number ratio \( r_f=2, 3, 4 \), we can group the cities into different classes according to the \( 2^n \) rule, \( 3^n \) rule, and \( 4^n \) rule.

<table>
<thead>
<tr>
<th>( m )</th>
<th>Total block ( (S_m) )</th>
<th>Average size ( (P_m) )</th>
<th>Number ( (f_m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20866</td>
<td>20866.0</td>
<td>1</td>
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<tr>
<td>2</td>
<td>50709</td>
<td>25354.5</td>
<td>2</td>
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<td>3</td>
<td>77576</td>
<td>19394.0</td>
<td>4</td>
</tr>
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<td>4</td>
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<td>16</td>
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<td>6</td>
<td>89012</td>
<td>2528.5</td>
<td>32</td>
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<td>64</td>
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<td>8</td>
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<td>79299</td>
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<td>256</td>
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<td>84527</td>
<td>164.7</td>
<td>512</td>
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<tr>
<td>11</td>
<td>84599</td>
<td>82.6</td>
<td>1024</td>
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<tr>
<td>12</td>
<td>75214</td>
<td>36.7</td>
<td>2048</td>
</tr>
<tr>
<td>13</td>
<td>21820</td>
<td>20.5</td>
<td>1065*</td>
</tr>
</tbody>
</table>
• For common ratio $N=2$, the scaling relation is

$$\hat{f}_m = 91161.315 P_m^{-1.025}$$

• The goodness of fit is about $R^2=0.996$, and the fractal dimension $D\approx1.025$. The average size ratio within the scaling range is about $r_p=1.942$, which is very close to $r_f=2$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>Total block ($S_m$)</th>
<th>Average size ($P_m$)</th>
<th>Number ($f_m$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>28866.0</td>
<td>1</td>
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<tr>
<td>2</td>
<td>72482</td>
<td>24160.7</td>
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<td>3</td>
<td>125128</td>
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<td>4</td>
<td>128817</td>
<td>4771.0</td>
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</tr>
<tr>
<td>5</td>
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</tr>
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<td>132372</td>
<td>181.6</td>
<td>729</td>
</tr>
<tr>
<td>8</td>
<td>129061</td>
<td>59.0</td>
<td>2187</td>
</tr>
<tr>
<td>9</td>
<td>44397</td>
<td>23.6</td>
<td>1880*</td>
</tr>
</tbody>
</table>

• Number ratio $r_f=3$.

• For $N=3$, the scaling relation is

$$\hat{f}_m = 173887.792 P_m^{-1.056}$$

• The goodness of fit is about $R^2=0.995$, and the fractal dimension is about $D=1.056$. The mean of size ratios within the scaling range is estimated as $r_p=2.779$, near $r_f=3$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>Total block ($S_m$)</th>
<th>Average size ($P_m$)</th>
<th>Number ($f_m$)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>10031.5</td>
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<td>4</td>
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<td>581.1</td>
<td>256</td>
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<tr>
<td>6</td>
<td>167362</td>
<td>163.4</td>
<td>1024</td>
</tr>
<tr>
<td>7</td>
<td>145868</td>
<td>38.4</td>
<td>3795*</td>
</tr>
</tbody>
</table>
For $N=4$, the scaling relation is

$$f_m = 195996.696 P_m^{-1.045}$$

The goodness of fit is about $R^2=0.995$, and the fractal dimension is $D\approx 1.045$. The average size ratio within the scaling range is about $r_p=3.692$, close to $r_f=4$.

According to mathematical derivation, the top class is an outlier because the largest city always struggles between the rank-size rule and hierarchical scaling law.

The bottom class is also an outlier because of absence of city size data or undergrowth of cites and towns.

The last class of each hierarchy is a *lame-duck class* termed by Davis (1978), who first presented the $2^n$ rule of cities.


The natural cities of Germany lend further support to the equivalent relationship between the rank-size distribution and the self-similar hierarchy.

The same conclusion can be drawn from the data sets about the cities of France and UK.

The last class of each hierarchy is a *lame-duck class* termed by Davis (1978), who first presented the $2^n$ rule of cities.


The functions or uses are:

- Rescaling the rank-size law.
- Reconstructing geographical space.
- Reveal geographical mathematical laws.
- Reckoning the parameter values of geographical models.
Example 1—Central place fractals

- Central place theory is one of the cornerstones of human geography, and "our understanding of the growth and evolution of urban settlement systems largely rests upon the edifice of central place theory and its elaboration and empirical testing through spatial statistics" (Longley et al., 1991).


Example 1—Central place fractals

- Using the method of hierarchical rescaling, we can turn the pure theory of central places into an empirical theory, which can be testified by observational data.

Example 1—Central place fractals

- Central place systems of W. Christaller (1933/1966).

Example 1—Central place fractals

- Central place model ↔ Koch snowflake fractal

Example 1—Central place fractals

- System of urban places in Henan, China
  - The system consists of 38 cities in 2000 (census data).
Hierarchy of urban places in Henan

- The system of cities can be reorganized as a hierarchy with a cascade structure consisting of 6 classes and 38 cities.

Spatial disaggregation

- Hierarchical scaling ↔ Rescaling ↔ Dummy network

Dummy network

- Hierarchical scaling relation—the rank-size rescaling of cities.
- This scaling law is equivalent to Zipf’s law.

Datasets.

<table>
<thead>
<tr>
<th>m</th>
<th>$L_m$</th>
<th>$P_m$</th>
<th>$f_m$</th>
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<td>9.4227</td>
<td>91346.4286</td>
<td>7</td>
</tr>
</tbody>
</table>

Dummy network

- The population distribution scaling of cities.
- The hollow square is an outlier indicating the lame-duck class.
**Dummy network**

- The network structure scaling of cities. This is the core model for fractal central places.

**Dummy network (*)**

- The computational results of central place fractal.

<table>
<thead>
<tr>
<th>Relation</th>
<th>Model</th>
<th>Scaling exponent</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rank-size rescaling</td>
<td>$f_r = 5.995411 \times 10^{-0.6052}$</td>
<td>$D_r = 1.0622$</td>
<td>0.9819</td>
</tr>
<tr>
<td>Population distribution</td>
<td>$P_m = 8.671.8074 \times 10^{-1.8093}$</td>
<td>$D_p = 1.8093$</td>
<td>0.9821</td>
</tr>
<tr>
<td>Network structure</td>
<td>$f_m = 3.98.711e_6 \times 10^{-3.933}$</td>
<td>$D_f = 1.9192$</td>
<td>0.9616</td>
</tr>
</tbody>
</table>

**Example 1—Central place fractals**

- If we change $r_f = 2$ to $r_f = 3$ or $r_f = 4$ or $r_f = 5$, we can testify the spatial patterns of central place system.

**Example 2—Distance friction coefficient**

- The geographical gravity model (GGM) is one of very important spatial interaction models in geography.
- The form familiar to us is as follows.

**Example 2—Distance friction coefficient**

- The key is how to estimate the distance friction coefficient, $b$.
- Geographers used to employ the linear regression to estimate the $b$ value.
- In fact, the hierarchical rescaling method is a simple approach to estimating the average value of $b$ in a region.
Example 2—Distance friction coefficient

- By hierarchical rescaling, we can derive a relation such as

\[ b = qD_f = \frac{D_f}{D_p} \]

- \( b \) – distance friction coefficient
- \( q \) – Zipf’s exponent
- \( D_f \) – fractal dimension of cities
- \( D_p \) – fractal dimension of urban population

Example 2—Distance friction coefficient

- Let’s take the cities of Henan Province, China as an example.
- Partial parameters have been estimated by hierarchical scaling and rescaling. (**)

Example 2—Distance friction coefficient

- The rank-size patterns of Henan’s cities.
- \( q \approx 0.9677 \).
- \( b \approx 1.8093 \).
- On the other hand, \( q \approx 0.9677 \).
- So, we have \( b = qD_f \approx 0.9677 \times 1.9192 \approx 1.8572 \).
- This suggests that, approximately, the \( b \) value come between 1.8 and 1.86.

Example 3—Allometric growth

- The hierarchical scaling and rescaling can be used to reveal the allometric relation properly and estimate the allometric scaling exponent.

Example 3—Allometric growth

- The theoretical relations between the rank-size distribution and allometry are as follows.

<table>
<thead>
<tr>
<th>Case</th>
<th>Rank-size distribution</th>
<th>Allometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Linear relation</td>
<td>Linear relation</td>
</tr>
<tr>
<td>2</td>
<td>Logarithmic relation</td>
<td>Linear relation</td>
</tr>
<tr>
<td>3</td>
<td>Exponential relation</td>
<td>Power law</td>
</tr>
<tr>
<td>4</td>
<td>Power law</td>
<td>Power law</td>
</tr>
<tr>
<td>5</td>
<td>Lognormal relation</td>
<td>Power law</td>
</tr>
</tbody>
</table>
Example 3—Allometric growth

- However, in practice, the case are complicated.
- For example, for the cities of the United States of America (USA), the allometric scaling relation cannot be properly revealed by using traditional methods.

The datasets come from Bin Jiang and his coworkers.

The measures are as below:

- Size—street junction number.
- Area—areal extent.

The rank-size distribution of junctions and areal extent.

However, the allometry seems to take on a linear relation rather than following the power law.

The allometric relation is inconsistent with Zipf’s law.

If we employ the hierarchical rescaling method, the power-law relation will emerge.
Example 3—Allometric growth

• The allometric scaling relation between junction number, \( P_m \), and areal extent, \( A_m \).

\[
A_m = nP_m^\sigma \\
= 79105 P_m^{0.9427} \\
\sigma = 0.9427 \\
R^2 = 0.9995
\]

Example 3—Allometric growth

• The hierarchical rescaling can provide at least three approaches to estimating the allometric scaling exponent.
  1. The power-law method.
  2. The exponential-law method.
  3. The common ratio method.

Brief summary

• The hierarchical scaling and rescaling (HSR) — A theory.
• HSR — A method for scaling analysis.
• HSR — A transform from dilation symmetry to translational symmetry.
• HSR — A mathematical link between simplicity and complexity.

Thanks!